

Generalized conditions for genuine multipartite continuous-variable entanglement

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We derive a hierarchy of continuous-variable multipartite entanglement conditions in terms of second-order moments of position and momentum operators that generalizes existing criteria. Each condition corresponds to a convex optimization problem which, given the covariance matrix of the state, can be numerically solved in a straightforward way. The conditions are independent of partial transposition and thus are also able to detect bound entangled states. Our approach can be used to obtain an analytical condition for genuine multipartite entanglement. We demonstrate that even a special case of our conditions can detect entanglement very efficiently. Using multi-objective optimization it is also possible to numerically verify genuine entanglement of some experimentally realizable states.

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I. INTRODUCTION

In the multipartite case there are many different notions of entanglement, ranging from the most specific to the most general ones. A specific kind of entanglement means that we precisely specify the groups of parties that are separable from each other, i.e., we specify a partition of the set of indices $\{1, \dots, n\}$. A partition $\mathcal{I} = \{I_1, \dots, I_k\}$ is a disjoint set of nonempty subsets of the indices whose union is equal to $\{1, \dots, n\}$. For example, in the case $n = 4$ there are partitions $\{2\} \cup \{1, 3, 4\}$, $\{1\} \cup \{2\} \cup \{3, 4\}$ and many others. We will use a more compact notation for partitions and write them as $2|134$ and $1|2|34$. A partition \mathcal{I}' is finer than a partition \mathcal{I} (or the partition \mathcal{I} is coarser than \mathcal{I}') if for any I'_i there is an I_j such that $I'_i \subseteq I_j$. For example, the partition $1|2|34$ is finer than the partition $12|34$. There are two extreme partitions, the trivial partition $12\dots n$ and the partition $1|2|\dots|n$. The former corresponds to the case where no information about separability properties is known and the latter corresponds to the notion of full separability, where all parties are separable from each other. The set of partitions with the finer-than relation is referred to as partition lattice and can be visualized as the graph shown in Fig. 1 for $n = 4$. At the bottom is the trivial partition $1\dots n$, then on the second line are all $2^{n-1} - 1$ partitions into two parts, next are all partitions into three parts, and so on. At the top is the partition $1|2|\dots|n$. Partitions with $k = 2$ play a special role and are called bipartitions.

From these specific kinds of separability one can construct more general types by considering mixtures of specific kinds according to some criteria. For example, a multipartite state is called k -separable, $k \geq 2$, if it is a mixture of states each of which has k separable groups of modes (the corresponding partitions are on

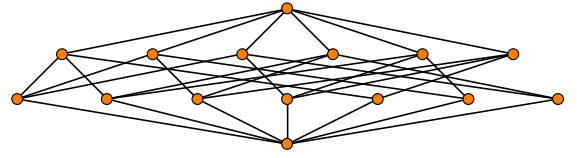


FIG. 1. (Color online). The partition lattice of four parties. Nodes are partitions and edges correspond to the "finer than" relation.

the same line in the lattice diagram). In the case of $n = 4$, a 3-separable state is a state which is a mixture of $1|2|34$ -, $1|3|24$ -, $1|4|23$ -, $2|3|14$ -, $13|2|4$ - and $12|3|4$ -separable states. The notion of n -separability of an n -partite state is the same as the notion of full separability. A 2-separable state is referred to as biseparable and the notion of biseparability is the most general of all — a state which is separable in any sense discussed above is automatically biseparable. On the other hand, full separability is the most specific kind of separability — a fully separable state is separable in any other sense.

In general, a biseparable state is a mixture of $2^{n-1} - 1$ states corresponding to all nontrivial bipartitions of n parties. There are states that cannot be represented as such a mixture. These states, the states that are not biseparable, are called genuine multipartite entangled. The problem of recognizing which states are genuine entangled is very important for different applications. It is a highly nontrivial task to develop practical conditions for detecting genuine multipartite entanglement. In the following a single party always corresponds to a single electromagnetic mode as known from continuous-variable (CV) quantum information theory [1, 2]. Many such conditions for n -partite CV systems deal with lower bounds for the second-order quantity

$$\mathcal{G} = \langle \hat{\mathbf{r}}^T M \hat{\mathbf{r}} \rangle = \sum_{i,j=1}^n M_{ij} \langle \{\hat{r}_i, \hat{r}_j\} \rangle = \text{Tr}(M\gamma), \quad (1)$$

where $\hat{\mathbf{r}} = (\hat{\mathbf{x}}, \hat{\mathbf{p}})$ is the $2n$ -vector of position and

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momentum operators and $M = (M_{ij})_{i,j=1}^{2n}$ is a real, symmetric, positive definite $2n \times 2n$ matrix and $\gamma = (\langle \{\hat{r}_i, \hat{r}_j\} \rangle)_{i,j=1}^{2n}$ is the covariance matrix of the state, $\langle \{\hat{A}, \hat{B}\} \rangle = (1/2) \langle \hat{A}\hat{B} + \hat{B}\hat{A} \rangle$. In fact, a special case of matrix M with zero off-diagonal blocks is usually used in practice. Denoting $X = (M_{ij})_{i,j=1}^n$ and $P = (M_{ij})_{i,j=n+1}^{2n}$, which are symmetric, positive semidefinite $n \times n$ matrices, the quantity \mathcal{G} in this case reads as

$$\mathcal{G} = \langle \hat{\mathbf{x}}^T X \hat{\mathbf{x}} + \hat{\mathbf{p}}^T P \hat{\mathbf{p}} \rangle = \text{Tr}(X\gamma_{xx}) + \text{Tr}(P\gamma_{pp}), \quad (2)$$

where $\gamma_{xx} = (\langle \hat{x}_i \hat{x}_j \rangle)_{i,j=1}^n$ and $\gamma_{pp} = (\langle \hat{p}_i \hat{p}_j \rangle)_{i,j=1}^n$ are the diagonal blocks of the covariance matrix of the state. For example, the classical result of Ref. [3] uses rank-one matrices $X = \begin{pmatrix} a^2 & 1 \\ 1 & a^{-2} \end{pmatrix} = \mathbf{h}\mathbf{h}^T$ and $P = \begin{pmatrix} a^2 & -1 \\ -1 & a^{-2} \end{pmatrix} = \mathbf{g}\mathbf{g}^T$, where $\mathbf{h}^T = (a, a^{-1})$ and $\mathbf{g}^T = (a, -a^{-1})$. The works [4, 5] use general rank-one matrices $X = \mathbf{h}\mathbf{h}^T$ and $P = \mathbf{g}\mathbf{g}^T$, where \mathbf{h} and \mathbf{g} are real n -vectors. In Ref. [6] the quantity \mathcal{G} with 3×3 matrices was used. Refs. [7, 8] deal with the general quantity \mathcal{G} . Among the works that also use second order moments we mention [9–11]. As based on second-order moments only, all these criteria are sufficient entanglement witnesses for all CV states, but necessary and sufficient only for Gaussian states.

In a realistic setting errors of measurements are unavoidable, so to be completely rigorous we need to incorporate the possibility of errors into our scheme. We do this and formulate a hierarchy of entanglement conditions as convex optimization problems for \mathcal{G} in terms of the covariance matrix γ and the information about errors of the measurements. For discrete variables a convex optimization approach has been developed in Ref. [12].

The paper is organized as follows. In section II we obtain the minimal value of \mathcal{G} defined by Eq. (2) over all quantum states. In Section III we show how to improve the lower bound obtained in the preceding section for separable states and construct a hierarchy of separability conditions in the form of convex optimization problems. In Section IV we apply our construction to rank one matrices and demonstrate that it coincides with some previously known results, so that those results are just a special case of our more general approach. In Section V we give an example of an entangled state with positive partial transposition that can be detected by our conditions. Section VI is devoted to an analytical condition for genuine multipartite entangled that can be obtained from our hierarchy of conditions. In Section VII we further extend our approach by taking measurement errors into consideration and demonstrating that our method works in realistic settings as well. Then we give a conclusion and provide in appendices all technical details missing in the main part of our work.

II. QUANTUMNESS BOUND

First of all, we find the minimum of \mathcal{G} over all quantum states and then we show how this bound can be

improved for separable states. To find the minimal value of \mathcal{G} for a given matrix M we use the Williamson's theorem [13]. This theorem states that there is a symplectic matrix S such that $S^T M S = \begin{pmatrix} \Lambda & 0 \\ 0 & \Lambda \end{pmatrix}$, where Λ is a diagonal $n \times n$ matrix. Since each symplectic transform is implementable as a unitary transformation [14], starting with a state $\hat{\varrho}$ we have

$$\langle \hat{\mathbf{r}}^T M \hat{\mathbf{r}} \rangle = \left\langle \hat{\mathbf{r}}^T \begin{pmatrix} \Lambda & 0 \\ 0 & \Lambda \end{pmatrix} \hat{\mathbf{r}} \right\rangle' \quad (3)$$

for the appropriately transformed state $\hat{\varrho}'$. The minimum of \mathcal{G} is achieved for $\hat{\varrho}'$ being the vacuum state, and in this case the equality $\mathcal{G} = \text{Tr} \Lambda$ takes place. We thus obtain that the inequality $\mathcal{G} = \text{Tr}(M\gamma) \geq \text{Tr} \Lambda$ is valid for all quantum states and it is tight.

To compute the minimal value of \mathcal{G} we need to know the diagonal elements λ_j of Λ . These numbers are referred to as *symplectic spectrum* of M and they can be directly obtained from the matrix M according to the fact that $\pm i\lambda_j$ are the eigenvalues of JM , where $J = \begin{pmatrix} 0 & E \\ -E & 0 \end{pmatrix}$, and E is the identity matrix. In our special case of block-diagonal $M = \begin{pmatrix} X & 0 \\ 0 & P \end{pmatrix}$ we can get these numbers directly in terms of X and P . In fact, we have the equality

$$JM = \begin{pmatrix} 0 & E \\ -E & 0 \end{pmatrix} \begin{pmatrix} X & 0 \\ 0 & P \end{pmatrix} = \begin{pmatrix} 0 & P \\ -X & 0 \end{pmatrix}. \quad (4)$$

The characteristic equation of this matrix is $\chi(\lambda) = \det \begin{pmatrix} \lambda E - P \\ X & \lambda E \end{pmatrix} = 0$. Since the diagonal blocks commute with the off-diagonal ones, according to [15] this equation can be simplified as $\chi(\lambda) = \det(\lambda^2 E + XP) = 0$. Substituting the eigenvalues $\pm i\lambda_j$ into this equation we see that the diagonal elements λ_j satisfy the equation

$$\det(XP - \lambda^2 E) = \det(\sqrt{X}P\sqrt{X} - \lambda^2 E) = 0, \quad (5)$$

from which it follows that they are the eigenvalues of the symmetric matrix $\sqrt{X}P\sqrt{X}$. The matrices X and P can be swapped in this derivation. We arrive to the following result: The minimal value of \mathcal{G} is given by the expression

$$\min_{\hat{\varrho}} \mathcal{G} = \text{Tr} \sqrt{\sqrt{X}P\sqrt{X}} = \text{Tr} \sqrt{\sqrt{P}X\sqrt{P}}. \quad (6)$$

To put it in another way, we have the tight inequality

$$\text{Tr}(X\gamma_{xx}) + \text{Tr}(P\gamma_{pp}) \geq \mathcal{B}(X, P) = \text{Tr} \sqrt{\sqrt{X}P\sqrt{X}}, \quad (7)$$

which is valid for all positive semidefinite matrices X and P and all correlation matrices γ_{xx} and γ_{pp} . This inequality gives some bound on \mathcal{G} . The bound has been obtained without any assumptions about separability properties of the state in question and thus is valid for all multipartite quantum states. Our task now is to improve this bound for separable states. The improvement will depend on the separability kind of the state — the more separable groups of modes the state has, the higher is its separability kind in the partition lattice and the more can this

estimation be improved. In Appendix A we also show that from the inequality (7) one can get a special case of Araki-Lieb-Thirring trace inequalities [16–18]. Such inequalities play some role in quantum entropy theory, see Ref. [19].

III. HIERARCHY OF SEPARABILITY CONDITIONS

We first show that pure states with real wave functions are enough to minimize $\mathcal{G} = \mathcal{G}(X, P)$. In fact, in terms of wave functions the quantity \mathcal{G} reads as

$$\mathcal{G} = \int \left((\mathbf{x}^T X \mathbf{x}) |\psi(\mathbf{x})|^2 + (\nabla \psi^*)^T P (\nabla \psi) \right) d\mathbf{x}. \quad (8)$$

If we take a general wave function of the form $\psi(\mathbf{x}) = f(\mathbf{x})e^{i\varphi(\mathbf{x})}$, where $f(\mathbf{x}) = |\psi(\mathbf{x})|$ is a real wave function and $\varphi(\mathbf{x})$ is the phase, we will see that the first term in Eq. (8) does not depend on $\varphi(\mathbf{x})$, while the other term is equal to

$$\begin{aligned} & \int ((\nabla f)^T P (\nabla f) + f^2 (\nabla \varphi)^T P (\nabla \varphi)) d\mathbf{x} \\ & \geq \int (\nabla f)^T P (\nabla f) d\mathbf{x}. \end{aligned} \quad (9)$$

It follows that for any wave function $\psi(\mathbf{x})$ there is a real wave function $f(\mathbf{x})$ (the absolute value of the former) such that $\mathcal{G}_\psi(X, P) \geq \mathcal{G}_f(X, P)$ and thus we can consider only pure states with real wave functions.

If a pure state with real wave function is $\{I_1, \dots, I_k\}$ -separable then $\gamma_{pp,ij} = \langle \hat{p}_i \hat{p}_j \rangle = 0$ if the indices i and j belong to different blocks I_l . In fact, separability for pure states means that the wave function is factorizable, i.e., $\psi(\mathbf{x}) = f(\mathbf{x}')g(\mathbf{x}'')$, where, without loss of generality, we can assume that $\mathbf{x}' = (x_1, \dots, x_k)$, $\mathbf{x}'' = (x_{k+1}, \dots, x_n)$. For $1 \leq i \leq k$ and $k+1 \leq j \leq n$ we have

$$\begin{aligned} \langle \hat{p}_i \hat{p}_j \rangle &= \int \frac{\partial \psi}{\partial x_i}(\mathbf{x}) \frac{\partial \psi}{\partial x_j}(\mathbf{x}) d\mathbf{x} \\ &= \int f(\mathbf{x}') \frac{\partial f}{\partial x_i}(\mathbf{x}') d\mathbf{x}' \int g(\mathbf{x}'') \frac{\partial g}{\partial x_j}(\mathbf{x}'') d\mathbf{x}'' = 0. \end{aligned} \quad (10)$$

The fact that $\psi(\mathbf{x})$ is real is important for the validity of the last step. The seeming asymmetry in position and momentum operators is superficial — if we worked in momentum representation and dealt with real wave function in that representation we would have $\langle \hat{x}_i \hat{x}_j \rangle = 0$.

We have just shown that $\langle \hat{p}_i \hat{p}_j \rangle = 0$ if modes i and j are separable, but for the position correlations we can say only that they factorize: $\langle \hat{x}_i \hat{x}_j \rangle = \langle \hat{x}_i \rangle \langle \hat{x}_j \rangle$. We can get a similar conclusion for position moments if we take minimization property into account. If a state $|\psi\rangle$ minimizes \mathcal{G} then we can also assume that $\langle \hat{x}_i \hat{x}_j \rangle = 0$. In fact, taking the wave function $\psi_0(\mathbf{x}) = \psi(\mathbf{x} + \mathbf{x}_0)$, where $\mathbf{x}_0 = \langle \mathbf{x} \rangle$ is the vector of averages computed for the wave function $\psi(\mathbf{x})$, we get a new wave function with $\langle \mathbf{x} \rangle_0 = \mathbf{0}$ and

$\mathcal{G}_0 = \mathcal{G} - \langle \mathbf{x} \rangle^T X \langle \mathbf{x} \rangle \leq \mathcal{G}$. If X is positive definite then we must have $\langle \mathbf{x} \rangle = \mathbf{0}$ and thus $\langle \hat{x}_i \hat{x}_j \rangle = 0$ for any separable state $\psi(\mathbf{x})$ that minimizes \mathcal{G} . If X is degenerate then we can just find a separable state with $\langle \hat{x}_i \hat{x}_j \rangle = 0$ that minimizes \mathcal{G} , but there can be minimizing states that do not have this property. Moreover, this new wave function $\psi_0(\mathbf{x})$ also has the property that $\langle \hat{p}_i \hat{p}_j \rangle = 0$ if i and j are separable. We see that among $\{I_1, \dots, I_k\}$ -separable states minimizing \mathcal{G} we can always find a pure state with real wave function for which $\gamma_{xx}[I_i|I_j] = \gamma_{pp}[I_i|I_j] = 0$ for $1 \leq i, j \leq k$, $i \neq j$. The notation $A[I|I']$, where I and I' are sets of indices, is used to denote the submatrix of A formed by the intersection of rows with indices in I and columns with indices in I' . Using such a state, we can improve the lower bound for \mathcal{G} .

Due to the relations $\langle \hat{x}_i \hat{x}_j \rangle = \langle \hat{p}_i \hat{p}_j \rangle = 0$ for separable i and j we have the equality

$$\text{Tr}(X\gamma_{xx}) + \text{Tr}(P\gamma_{pp}) = \text{Tr}(X_{\mathbf{u}}\gamma_{xx}) + \text{Tr}(P_{\mathbf{v}}\gamma_{pp}), \quad (11)$$

where $X_{\mathbf{u}}$ is obtained from $X = (x_{ij})_{i,j=1}^n$ by replacing its elements corresponding to zero elements of γ_{xx} by arbitrary real numbers u_i subject to the condition that $X_{\mathbf{u}}$ is symmetric and positive semidefinite and the same procedure is applied to P to produce $P_{\mathbf{v}}$. In other words, we can replace all elements of the submatrices of the form $X[I_i|I_j]$ and $P[I_i|I_j]$, $i \neq j$, by arbitrary real numbers in such a way that the resulting matrices are again symmetric and positive semidefinite. This construction is better illustrated by an example. For $n = 4$, consider $2|134$ - and $1|2|34$ -factorizable states. In the former case we have $I_1 = \{2\}$, $I_2 = \{1, 3, 4\}$, and in the latter case we have $I_1 = \{1\}$, $I_2 = \{2\}$, $I_3 = \{3, 4\}$. The matrices $X_{\mathbf{u}}$ corresponding to these two cases are

$$\begin{pmatrix} x_{11} & u_1 & x_{13} & x_{14} \\ u_1 & x_{22} & u_2 & u_3 \\ x_{13} & u_2 & x_{33} & x_{34} \\ x_{14} & u_3 & x_{34} & x_{44} \end{pmatrix}, \quad \begin{pmatrix} x_{11} & u_1 & u_2 & u_3 \\ u_1 & x_{22} & u_4 & u_5 \\ u_2 & u_4 & x_{33} & x_{34} \\ u_3 & u_5 & x_{34} & x_{44} \end{pmatrix}. \quad (12)$$

In the first case we replace elements of the submatrices $X[I_1|I_2]$ and $X[I_2|I_1]$ by arbitrary numbers, and in the second case we replace submatrices $X[I_1|I_2]$, $X[I_2|I_1]$, $X[I_1|I_3]$, $X[I_3|I_1]$, $X[I_2|I_3]$, $X[I_3|I_2]$. Different submatrices are marked by different colors (symmetric parts are marked by the same color). The more factorizable parts the state has the more elements can be replaced by arbitrary numbers. For a completely factorizable state we can freely choose all the off-diagonal entries.

In order not to overload the notation, we fix some kind of separability, i.e., some decomposition $\mathcal{I} = \{I_1, \dots, I_k\}$ of the indices, and use it in all the considerations below. Applying the inequality (7), we find that for a pure \mathcal{I} -factorizable state with real wave function, and thus for all \mathcal{I} -separable states, we have

$$\mathcal{G}(X, P) \geq \mathcal{B}_{\mathcal{I}}(X, P) = \max_{\mathbf{u}, \mathbf{v}} \text{Tr} \sqrt{\sqrt{X_{\mathbf{u}}} P_{\mathbf{v}} \sqrt{X_{\mathbf{u}}}}. \quad (13)$$

where the optimization is over the points \mathbf{u} and \mathbf{v} such that $X_{\mathbf{u}}$ and $P_{\mathbf{v}}$ are positive definite. For example, for a bipartition $\{I_1, I_2\}$ with $|I_1| = m$, $|I_2| = n - m$ the vectors \mathbf{u} and \mathbf{v} have $m(n - m)$ components, so in this case the optimization problem has $2m(n - m)$ variables. The full separability optimization problem has $n(n - 1)$ components.

Note that for the vectors \mathbf{u}_0 and \mathbf{v}_0 with the corresponding elements of the matrices X and P , respectively, we have $X_{\mathbf{u}_0} = X$ and $P_{\mathbf{v}_0} = P$ and thus for any partition $\mathcal{B}_{\mathcal{I}}(X, P) \geq \mathcal{B}(X, P)$. This inequality is strict for most of the matrices X and P . It follows that the inequality (13) gives a stronger bound on \mathcal{G} than does the inequality (7). Moreover, in the same way we obtain that if \mathcal{I}' is finer than \mathcal{I} then $\mathcal{B}_{\mathcal{I}'}(X, P) \geq \mathcal{B}_{\mathcal{I}}(X, P)$, since in this case we have more variables to optimize over, and thus the condition for \mathcal{I}' -separable states gives a stronger bound than the condition for \mathcal{I} -separability. The condition for full separability gives the strongest bound of all. We have just obtained a hierarchy of separability conditions that mirrors the partition lattice in Fig. 1.

To each kind of separability \mathcal{I} corresponds its own maximization problem of its own dimension. There are $2^{n-1} - 1$ different bipartitions of the indices of an n -partite state and many more partitions into three or more parts. If for a given state there is a pair of matrices X and P such that an inequality (13) is violated, then the state is not \mathcal{I} -separable. If there are X and P such that the inequalities (13) are violated for all bipartitions simultaneously, then the state is genuine multipartite entangled.

IV. RANK-ONE MATRICES

We now show that the results of Ref. [4] are just a special case of the general inequalities (13) and give an analytical solution of these optimization problems in this special case. Consider the rank-one matrices $X = \mathbf{h}\mathbf{h}^T$ and $P = \mathbf{g}\mathbf{g}^T$. The square root of a rank-one matrix $A = \mathbf{a}\mathbf{a}^T$ is given by $\sqrt{A} = \mathbf{a}\mathbf{a}^T / \|\mathbf{a}\|$, so we have $\mathcal{B}(X, P) = |(\mathbf{h}, \mathbf{g})|$. As a concrete example let us consider four-partite case and $\mathcal{I} = 1|2|3|4$ -separable states. We are free to change some elements of the matrices X and P . Let us just change the sign of the X and P 's elements that are marked in Eq. (12):

$$X' = \begin{pmatrix} h_1^2 & \pm h_1 h_2 & \pm h_1 h_3 & \pm h_1 h_4 \\ \pm h_1 h_2 & h_2^2 & \pm h_2 h_3 & \pm h_2 h_4 \\ \pm h_1 h_3 & \pm h_2 h_3 & h_3^2 & h_3 h_4 \\ \pm h_1 h_4 & \pm h_2 h_4 & h_3 h_4 & h_4^2 \end{pmatrix}. \quad (14)$$

For appropriate combinations of signs we can get that $X' = \mathbf{h}'\mathbf{h}'^T$ and $P' = \mathbf{g}'\mathbf{g}'^T$, where the new vectors read as $\mathbf{h}' = (\pm h_1, \pm h_2, h_3, h_4)$ and $\mathbf{g}' = (\pm g_1, \pm g_2, g_3, g_4)$, and thus

$$\begin{aligned} \mathcal{B}_{\mathcal{I}}(X, P) &\geq \max |(\mathbf{h}', \mathbf{g}')| \\ &= \max |\pm h_1 g_1 \pm h_2 g_2 + h_3 g_3 + h_4 g_4| \\ &= |h_1 g_1| + |h_2 g_2| + |h_3 g_3 + h_4 g_4| \geq \mathcal{B}(X, P). \end{aligned} \quad (15)$$

This result can be extended to all n and arbitrary kind of separability and coincides with the results obtained in Ref. [4].

Note that the inequality $\mathbf{h}^T \gamma_{xx} \mathbf{h} + \mathbf{g}^T \gamma_{pp} \mathbf{g} \geq |(\mathbf{h}, \mathbf{g})|$ is equivalent to the inequalities

$$\begin{pmatrix} \gamma_{xx} & \frac{1}{2}E \\ \frac{1}{2}E & \gamma_{pp} \end{pmatrix} \geq 0. \quad (16)$$

The same idea can be applied to the separability conditions and we arrive to the conditions for \mathcal{I} -separable states

$$\begin{pmatrix} \gamma_{xx} & \frac{1}{2}E_{\mathcal{I}} \\ \frac{1}{2}E_{\mathcal{I}} & \gamma_{pp} \end{pmatrix} \geq 0, \quad (17)$$

where $E_{\mathcal{I}}$ is a diagonal matrix with diagonal elements equal to ± 1 so that the elements with indices in the same group I_j have the same sign. These conditions work in many cases, but sometimes the more general conditions are needed. In the next section we give an example of a PPT state that satisfies the inequalities (17) but violates the general inequalities (13).

V. AN EXAMPLE OF PPT STATE

Consider a four-partite state with the covariance matrix given by [9, 20]

$$\begin{aligned} \gamma_{xx} &= \frac{1}{2} \begin{pmatrix} 2 & 0 & 1 & 0 \\ 0 & 2 & 0 & -1 \\ 1 & 0 & 2 & 0 \\ 0 & -1 & 0 & 2 \end{pmatrix}, \\ \gamma_{pp} &= \frac{1}{2} \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 4 & 0 \\ -1 & 0 & 0 & 4 \end{pmatrix}. \end{aligned} \quad (18)$$

The matrix $\gamma = \begin{pmatrix} \gamma_{xx} & 0 \\ 0 & \gamma_{pp} \end{pmatrix}$ is a covariance matrix of a quantum state, since $\gamma + (i/2)J \geq 0$. Partial transpositions of (1, 3)-kinds (that is transpositions 1|234, 2|134, 3|124 and 4|123) are negative, and these negativities are easily detected by the conditions (17). For example, the matrix $E_{1|234}$ reads as

$$E_{1|234} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad (19)$$

and the matrix $\begin{pmatrix} \gamma_{xx} & E_{1|234}/2 \\ E_{1|234}/2 & \gamma_{pp} \end{pmatrix}$ has negative eigenvalues. On the other hand, the partial transpositions of (2, 2)-kinds are all positive and the matrices (17) with $\mathcal{I} = 12|34, 13|24, 14|23$ are positive semidefinite, so the

test (17) does not detect any entanglement of a (2,2)-kind. Nevertheless, the state (18) is entangled in any of this kind.

To demonstrate this, consider the positive semidefinite matrices

$$X = \begin{pmatrix} x & 0 & -\sqrt{xy} & 0 \\ 0 & x & 0 & \sqrt{xy} \\ -\sqrt{xy} & 0 & y & 0 \\ 0 & \sqrt{xy} & 0 & y \end{pmatrix} \quad (20)$$

$$P = \begin{pmatrix} p & 0 & 0 & \sqrt{pq} \\ 0 & p & \sqrt{pq} & 0 \\ 0 & \sqrt{pq} & q & 0 \\ \sqrt{pq} & 0 & 0 & q \end{pmatrix},$$

where x, y, p and q are some positive numbers to be determined. The quantity $\mathcal{G}(X, P) = \text{Tr}(X\gamma_{xx}) + \text{Tr}(P\gamma_{pp})$ reads as $\mathcal{G}(X, P) = 2x + 2y - 2\sqrt{xy} + p + 4q - 2\sqrt{pq}$. To compute $\mathcal{B}_{12|34}(X, P)$, note that

$$\mathcal{B}_{12|34}(X, P) \geq \sqrt{\sqrt{X'}P'\sqrt{X'}}, \quad (21)$$

where the matrices X' and P' are defined to be (since we can play with the elements marked by color)

$$X' = \begin{pmatrix} x & 0 & \textcolor{brown}{0} & \textcolor{brown}{0} \\ 0 & x & \textcolor{brown}{0} & \textcolor{brown}{0} \\ \textcolor{brown}{0} & \textcolor{brown}{0} & y & 0 \\ \textcolor{brown}{0} & \textcolor{brown}{0} & 0 & y \end{pmatrix}, \quad P' = \begin{pmatrix} p & 0 & \textcolor{brown}{0} & \textcolor{brown}{0} \\ 0 & p & \textcolor{brown}{0} & \textcolor{brown}{0} \\ \textcolor{brown}{0} & \textcolor{brown}{0} & q & 0 \\ \textcolor{brown}{0} & \textcolor{brown}{0} & 0 & q \end{pmatrix}, \quad (22)$$

and thus for the boundary $\mathcal{B}_{12|34}$ we have the inequality

$$\mathcal{B}_{12|34} \geq 2(\sqrt{xp} + \sqrt{yq}). \quad (23)$$

It is easy to check that for the following values of the parameters:

$$\begin{aligned} x &= 0.144375, & y &= 0.084087, \\ p &= 0.232000, & q &= 0.039543 \end{aligned} \quad (24)$$

we have $\mathcal{G}(X, P) = 0.435170$, while according to Eq. (23) we have $\mathcal{B}_{12|34} \geq 0.481359$. We see that $\mathcal{B}_{12|34} > \mathcal{G}(X, P)$, and the PPT state Eq. (18) is 12|34-entangled. The construction for the partitions 13|24 and 14|23 is similar.

VI. GENUINE ENTANGLEMENT CONDITION

The inequalities (13) allow one to test states for entanglement of some kind, but the number of these kinds grows extremely fast with the number of parts. We now derive an analytical condition for genuine multipartite entanglement. It is a *single* condition that does not require testing exponentially many bipartitions, however,

it does not provide the best possible bound. Consider the quantity

$$\mathcal{G}_n = \sum_{1 \leq i < j \leq n} \langle (\hat{x}_i + \hat{x}_j)^2 + (\hat{p}_i - \hat{p}_j)^2 \rangle. \quad (25)$$

It is the general quantity \mathcal{G} with the following matrices $X = X_n$ and $P = P_n$:

$$X_n = \begin{pmatrix} n-1 & 1 & \dots & 1 & 1 \\ 1 & n-1 & \dots & 1 & 1 \\ \dots & \dots & \dots & \dots & \dots \\ 1 & 1 & \dots & n-1 & 1 \\ 1 & 1 & \dots & 1 & n-1 \end{pmatrix} \quad (26)$$

$$P_n = \begin{pmatrix} n-1 & -1 & \dots & -1 & -1 \\ -1 & n-1 & \dots & -1 & -1 \\ \dots & \dots & \dots & \dots & \dots \\ -1 & -1 & \dots & n-1 & -1 \\ -1 & -1 & \dots & -1 & n-1 \end{pmatrix}$$

Since matrices X_n and P_n are completely symmetric with respect to different parts, it is enough to consider only bipartitions of the form $1, \dots, k | k+1, \dots, n$ for $k \leq n/2$. We do not change the elements of X_n , and in the matrix P_n we set all v_i to 1 (so we change the matrix elements from -1 to 1). Denote the resulting matrix by $P'_{n,k}$. The matrices X_n and $P'_{n,k}$ commute, so that $\text{Tr}(\sqrt{X_n}P'_{n,k}\sqrt{X_n})^{1/2}$ is easy to compute

$$\text{Tr} \sqrt{X_n P'_{n,k}} = (n-1)\sqrt{n(n-2)} + 4\beta \frac{k(n-k)}{\sqrt{n}}, \quad (27)$$

where the number β is given by

$$\beta = \frac{1}{\sqrt{2n-2} + \sqrt{n-2}}. \quad (28)$$

The minimum of this expression over $1 \leq k \leq n/2$ is attained for $k = 1$. We have just obtained the following result: Any biseparable state must satisfy the inequality

$$\mathcal{G}_n \geq (n-1)\sqrt{n(n-2)} + \frac{4(n-1)}{\sqrt{n}(\sqrt{2n-2} + \sqrt{n-2})}. \quad (29)$$

If this inequality is violated, then the state is genuine multipartite entangled. Table I summarizes the lower bounds of \mathcal{G}_n for some n obtained with the analytical condition (29) and computed numerically from Eq. (13).

A similar genuine entanglement condition has been obtained in Ref. [4] in terms of rank-one matrices. The gap between quantum bound and biseparability bound there decreases as $O(1/n)$, where n is the number of parts[21]. In our condition this gap is $O(1)$ and so it does not tend to zero for large n .

VII. CONVEX OPTIMIZATION AND EXPERIMENTAL ERRORS

To demonstrate entanglement of a kind \mathcal{I} we need to find a pair of matrices X and P that violate the condition

n	2	3	4	5	6	7	8
q	0	3.46	8.48	15.49	24.49	35.49	48.49
a	2	5.00	10.03	17.06	26.07	37.08	50.09
b	2	5.46	10.89	18.26	27.59	38.89	52.17
f	2	6	12	20	30	42	56

TABLE I. Lower bounds for \mathcal{G}_n for different kinds of n -partite states. The q row is quantumness bound, $(n-1)\sqrt{n(n-2)}$. The a row is the biseparability bound given by the analytical expression (29). The b row is the true biseparability bound given by the solution of the optimization problem (13). The last row, f , is the full separability bound, $n(n-1)$.

(13). In practice, however, we should take into account the errors in the measurement of the matrix elements of γ_{xx} and γ_{pp} . Assuming that the errors in the individual elements of γ_{xx} and γ_{pp} are independent, the standard deviation $\sigma(X, P)$ of $\mathcal{G}(X, P)$ is given by the expression

$$\sigma^2(X, P) = \sum_{i,j=1}^n (x_{ij}^2 \sigma_{xx,ij}^2 + p_{ij}^2 \sigma_{pp,ij}^2), \quad (30)$$

where $\sigma_{xx,ij}$ and $\sigma_{pp,ij}$ are the standard deviations of individual elements of γ_{xx} and γ_{pp} respectively. So, to be on the safe side the right inequality reads as

$$\mathcal{E}_{\mathcal{I}}(X, P) = \mathcal{G}(X, P) + s \sigma(X, P) - \mathcal{B}_{\mathcal{I}}(X, P) \geq 0, \quad (31)$$

where s is the level of certainty with which we can claim that the state is entangled. We prove that the function $\mathcal{E}_{\mathcal{I}}(X, P)$ is convex on the set of all pairs of semidefinite matrices (X, P) .

To prove the convexity of $\mathcal{E}_{\mathcal{I}}(X, P)$ we have to prove the concavity of $\mathcal{B}_{\mathcal{I}}(X, P)$ because the convexity of the other two terms of $\mathcal{E}_{\mathcal{I}}(X, P)$ is obvious. The key element of this proof is the fact that $\text{Tr} \sqrt{\sqrt{X} \mathbf{u} P \mathbf{v} \sqrt{X} \mathbf{u}}$ is jointly concave with respect to all four variables X, P, \mathbf{u} and \mathbf{v} . Due to the equality

$$\text{Tr} \sqrt{\sqrt{X} P \sqrt{X}} = \min_{\gamma_{xx}, \gamma_{pp}} (\text{Tr}(X \gamma_{xx}) + \text{Tr}(P \gamma_{pp})) \quad (32)$$

it immediately follows $\text{Tr} \sqrt{\sqrt{X} P \sqrt{X}}$ is concave with respect to X and P as the minimum of a family of concave (in fact, linear) functions. From the relation

$$\begin{aligned} &(\theta X_1 + (1-\theta)X_2)'(\theta \mathbf{u}_1 + (1-\theta)\mathbf{u}_2) \\ &= \theta X_1'(\mathbf{u}_1) + (1-\theta)X_2'(\mathbf{u}_2), \end{aligned} \quad (33)$$

$0 \leq \theta \leq 1$, and similar relation for P we derive the joint concavity of $\text{Tr} \sqrt{\sqrt{X} \mathbf{u} P \mathbf{v} \sqrt{X} \mathbf{u}}$.

We now have three sets — the set $\Omega_{x,1} \times \Omega_{p,1}$ of points (\mathbf{u}, \mathbf{v}) where $X_1'(\mathbf{u})$ and $P_1'(\mathbf{v})$ are positive definite, the similar set $\Omega_{x,2} \times \Omega_{p,2}$ for X_2 and P_2 , and the set $\Omega_x \times \Omega_p$ for $X = \theta X_1 + (1-\theta)X_2$ and $P = \theta P_1 + (1-\theta)P_2$. In general, these are distinct sets, but one can easily see that $\theta \Omega_{x,1} \times \Omega_{p,1} + (1-\theta) \Omega_{x,2} \times \Omega_{p,2} \subseteq \Omega_x \times \Omega_p$. The

standard argument given in Ref. [22] can be applied here to conclude that $\mathcal{B}_{\mathcal{I}}(X, P)$ is concave as the maximum of a jointly concave function over a convex set. This finishes the proof of the convexity of the function $\mathcal{E}_{\mathcal{I}}(X, P)$ defined by Eq. (31).

To violate the inequality (31) we thus have to optimize $\mathcal{E}_{\mathcal{I}}(X, P)$ over X and P and check whether the optimal value is negative or not. This optimization problem has $n(n+1)$ variables, the elements of X and P . Since the function $\mathcal{E}_{\mathcal{I}}(X, P)$ is homogeneous, $\mathcal{E}_{\mathcal{I}}(\lambda X, \lambda P) = \lambda \mathcal{E}_{\mathcal{I}}(X, P)$ for $\lambda \geq 0$, it makes sense to put some condition on the matrices X and P . The simplest is a linear condition, for example, the condition $\text{Tr}(X \gamma_{xx} + P \gamma_{pp}) = C$, where C is an arbitrary fixed positive constant. We thus arrive to the following \mathcal{I} -separability condition:

$$\min_{\text{Tr}(X \gamma_{xx} + P \gamma_{pp})=C} (s \sigma(X, P) - \mathcal{B}_{\mathcal{I}}(X, P)) \geq -C. \quad (34)$$

If, for given $\gamma_{xx}, \gamma_{pp}, \sigma_{xx}$ and σ_{pp} , this inequality is violated, i.e., if this minimum drops below $-C$ then the state in question is not separable of the corresponding kind. If these inequalities can be violated for all bipartitions simultaneously (by the same pair of matrices X and P), then the state is genuine multipartite entangled. The methods to solve convex optimization problems like the one given by Eq. (34) are discussed, e.g., in Ref. [22].

What s should we choose? Usually, the "three-sigma rule", $s = 3$ is applied [23]. To better understand what values of s in Eq. (31) are sufficient to guarantee that our results are correct we need to know the probability that the result of a measurement lies outside s sigma interval. For a Gaussian probability distribution with the mean μ and the standard deviation σ this probability is given by the expression

$$\mathbf{P}(s) = 1 - \int_{\mu-s\sigma}^{\mu+s\sigma} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx = 1 - \text{erf}\left(\frac{s}{\sqrt{2}}\right), \quad (35)$$

which depends only on s . This probability decreases very quickly as s growth. The order of values of $\mathbf{P}(s)$ for some s are shown in the table below. In many cases, the value

s	1	2	3	4	5	6	7	10
$\mathbf{P}(s)$	0.32	0.05	10^{-3}	10^{-4}	10^{-6}	10^{-9}	10^{-12}	10^{-23}

of $s = 3$ is sufficient (the three-sigma rule). For $s \geq 5$ this probability is negligible, and for $s \geq 6$ it is practically zero. Even if the real probability distribution is not perfectly Gaussian it is unlikely to have long tail, so Eq. (35) gives a reasonable estimate. Even if this estimate is wrong by several orders of magnitude, provided that we have verified violation with $s \geq 6$ we are still on the safe side. The larger s we set the larger the probability of the correct result is, but the more difficult it will be to find a violation with such s . From this table we can conclude that we should search for a violation with s not smaller than 3 and not larger than 6 — the event

of getting the right result outside of six sigma interval is practically impossible.

We see that if we can violate our inequalities with $s = 6$ then it practically guarantees that our conclusion is correct. In Appendix B we demonstrate the usefulness of our approach by applying to some four-, six- and ten-partite realistic states. We also demonstrate that the four-partite state in question is genuine multipartite entangled.

VIII. CONCLUSION

We have developed a method to test continuous-variable multipartite states for arbitrary kinds of entanglement. Our approach allows both numerical and analytical treatment. Numerically, it reduces to a convex optimization problem, which allows fast and accurate solution. We have shown that it is very efficient at detecting ordinary entanglement and can detect genuine multipartite entanglement in a reasonable amount of time. Analytically, it allows to reproduce (and thus generalize) some known results as well as to obtain an analytical genuine multipartite entanglement condition. With our approach we can easily obtain a trace-class inequality, which is difficult to get in a direct way.

Appendix A: Trace inequalities

If matrices X and P commute then the right-hand side of Eq. (6) reduces to $\text{Tr} \sqrt{X} \sqrt{P} = \text{Tr} \sqrt{P} \sqrt{X}$. This expression, $\text{Tr}(\sqrt{X} \sqrt{P})$, is a lower bound for \mathcal{G} independent of commutation properties of X and P . For a real wave function $f(\mathbf{x})$ we have the equality

$$\mathcal{G}(X, P) = \int (\mathbf{u}^T(\mathbf{x}) X \mathbf{u}(\mathbf{x}) + \mathbf{v}^T(\mathbf{x}) P \mathbf{v}(\mathbf{x})) d\mathbf{x}, \quad (\text{A1})$$

where the vector fields $\mathbf{u}(\mathbf{x})$ and $\mathbf{v}(\mathbf{x})$ are defined via $\mathbf{u}(\mathbf{x}) = f(\mathbf{x})\mathbf{x}$ and $\mathbf{v}(\mathbf{x}) = \nabla f(\mathbf{x})$. We can write this equality in a more compact form as

$$\mathcal{G}(X, P) = \int (\|\tilde{\mathbf{u}}(\mathbf{x})\|^2 + \|\tilde{\mathbf{v}}(\mathbf{x})\|^2) d\mathbf{x}, \quad (\text{A2})$$

where the new vector fields are defined via $\tilde{\mathbf{u}} = \sqrt{X}\mathbf{u}$ and $\tilde{\mathbf{v}} = \sqrt{P}\mathbf{v}$. Now we can estimate \mathcal{G} as follows:

$$\begin{aligned} \mathcal{G}(X, P) &\geq 2 \left| \int (\tilde{\mathbf{u}}(\mathbf{x}), \tilde{\mathbf{v}}(\mathbf{x})) d\mathbf{x} \right| \\ &= 2 \left| \int \mathbf{x}^T \sqrt{X} \sqrt{P} (\nabla f)(\mathbf{x}) f(\mathbf{x}) d\mathbf{x} \right|. \end{aligned} \quad (\text{A3})$$

From the relation

$$\int u_j(\mathbf{x}) v_k(\mathbf{x}) d\mathbf{x} = \int x_j f(\mathbf{x}) \frac{\partial f(\mathbf{x})}{\partial x_k} d\mathbf{x} = -\frac{1}{2} \delta_{jk}, \quad (\text{A4})$$

we get the inequality $\mathcal{G}(X, P) \geq \text{Tr}(\sqrt{X} \sqrt{P})$.

We thus have two lower bounds for \mathcal{G} — the tight one is given by the inequality (6) and the other one, not necessarily tight, have just been obtained with the help of Cauchy-Schwarz inequality. Since the tight bound is the best bound possible, we derive the following inequality for a pair of positive definite matrices X and P :

$$\text{Tr} \sqrt{\sqrt{X} P \sqrt{X}} \geq \text{Tr}(\sqrt{X} \sqrt{P}) = \text{Tr}(\sqrt[4]{X} \sqrt{P} \sqrt[4]{X}). \quad (\text{A5})$$

This inequality is a special case of Araki-Lieb-Thirring trace inequalities [16–18], which also have quantum mechanical background and read as

$$\text{Tr}(A^{1/2} B A^{1/2})^{rq} \geq \text{Tr}(A^{r/2} B^r A^{r/2})^q, \quad (\text{A6})$$

where A and B are arbitrary positive definite matrices, $q \geq 0$ and $0 \leq r \leq 1$. The case of $q = 1$ and $r = 1/2$ corresponds to the inequality (A5).

Appendix B: Application to realistic matrices

It happens that rank-one matrices work surprisingly well. Consider the four-partite state that was analyzed in Ref. [8]. It has the following covariance matrix:

$$\begin{aligned} \gamma_{xx} &= \begin{pmatrix} 1.09921 & 0.16092 & -0.17609 & -0.84831 \\ 0.16092 & 0.40938 & -0.16060 & -0.18963 \\ -0.17609 & -0.16060 & 0.46060 & 0.04319 \\ -0.84831 & -0.18963 & 0.04319 & 1.06419 \end{pmatrix} \\ \gamma_{pp} &= \begin{pmatrix} 1.09921 & 0.35533 & 0.36439 & 0.91386 \\ 0.35533 & 0.92282 & 0.57440 & 0.43388 \\ 0.36439 & 0.57440 & 1.04339 & 0.34868 \\ 0.91386 & 0.43388 & 0.34868 & 1.06419 \end{pmatrix} \end{aligned} \quad (\text{B1})$$

The standard deviation matrix reads as

$$\begin{aligned} \sigma_{xx} &= \begin{pmatrix} 0.00327 & 0.01041 & 0.00894 & 0.00647 \\ 0.01041 & 0.00822 & 0.01848 & 0.01899 \\ 0.00894 & 0.01848 & 0.00861 & 0.01345 \\ 0.00647 & 0.01899 & 0.01345 & 0.00549 \end{pmatrix} \\ \sigma_{pp} &= \begin{pmatrix} 0.00458 & 0.01009 & 0.02767 & 0.04289 \\ 0.01009 & 0.01023 & 0.02101 & 0.02085 \\ 0.02767 & 0.02101 & 0.01466 & 0.01955 \\ 0.04289 & 0.02085 & 0.01955 & 0.00455 \end{pmatrix} \end{aligned} \quad (\text{B2})$$

A very simple way to search for violation of the condition (31) is to randomly generate 4-vectors \mathbf{h} and \mathbf{g} and check whether this condition is violated by the rank-one matrices $X = \mathbf{h}\mathbf{h}^T$ and $P = \mathbf{g}\mathbf{g}^T$, and if it is, how strong the violation is. Then just record the maximal observed violation. As a measure of violation we use the quantity

$$s = \frac{\mathcal{B}_T(X, P) - \mathcal{G}(X, P)}{\sigma(X, P)}. \quad (\text{B3})$$

This approach requires only simple matrix algebra manipulations, which can be done very efficiently with tools like Intel Math Kernel Library. A simple parallel Fortran program has been written and run on a low-end 4-core desktop PC. The total time to test all seven possible bipartitions in this four-partite case is 4 minutes (using all four cores available). Table II compares our results with those obtained in Ref. [8]. We see that for the state under study our approach is superior to that of Ref. [8] (which uses a genetic algorithm to find the best violation), since it is simpler and gives better results, though, as we have mentioned before, from practical point of view all violations larger than 6 are of the same value.

Bipart.	Violation		\mathbf{h} \mathbf{g}
	Ref. [8]	4m	
1 234	20.93	26.48	(1.97, -0.01, 0.49, 1.88) (1.14, 0.18, -0.20, -1.03)
2 134	13.17	18.08	(-0.40, -1.99, -1.48, -0.74) (-0.15, -1.80, 1.24, 0.68)
3 124	11.21	16.10	(0.31, 1.93, 1.62, 0.32) (0.23, 1.65, -1.46, -0.19)
4 123	21.06	26.57	(1.77, 0.18, 0.44, 1.82) (-0.96, -0.18, -0.12, 1.21)
12 34	24.34	27.79	(1.91, 0.42, 0.75, 1.90) (-1.06, -0.60, 0.50, 1.19)
13 24	23.52	26.17	(-1.52, -0.17, -0.54, -1.58) (0.78, -0.09, 0.24, -0.97)
14 23	4.66	9.72	(0.22, 1.65, 0.18, 0.77) (0.28, 1.34, -0.37, -0.95)

TABLE II. The comparison of the violation of the separability condition for the state given by the covariance matrix (B1) with measurements errors given by Eq. (B2). The second column shows the results obtained in Ref. [8], the third column lists the results obtained by randomly testing the inequalities (31). The total time to perform all seven tests is 4 minutes.

We now apply our technique to the six-partite state also considered in Ref. [8]. We have performed two runs of our program on the same hardware as in the previous case, one with a smaller number of random trials and the other with 200 times more trials. The first run takes approximately 4 minutes to perform all 31 tests, the other one takes around 12 hours. As Table. III demonstrates, in this case the optimization based on a genetic algorithm gives somewhat larger violations. On the other hand, we do not know what computational resources were used to perform that optimization and how much time it took. As we have already said, all violation larger than 6 are of the same practical value and our method produced much better violations in just a few minutes on a low-end PC.

The last state considered in Ref. [8] is a ten-partite state. It has been reported that the smallest violation of 1.1 was obtained for the bipartition 1,10|23456789. The corresponding probability to get wrong result is $\mathbf{P}(1.1) \approx 0.27$, and it is not small enough to conclude that the state under study is not 1,10|23456789-separable. Randomly generating vectors \mathbf{h} and \mathbf{g} , we have found that the inequality (31) for this kind of separability can

Bipartition	Violation		
	Ref. [8]	12h	4m
1 23456	40.086	40.111	37.500
2 13456	36.185	34.097	33.967
3 12456	20.274	19.715	17.683
4 12356	20.010	18.869	16.953
5 12346	27.146	26.680	22.979
6 12345	49.220	48.077	44.187
12 3456	53.541	53.085	50.142
13 2456	45.569	45.958	41.684
14 2356	44.789	42.163	42.509
15 2346	45.282	40.410	38.565
16 2345	31.177	29.995	25.686
23 1456	40.158	38.636	37.462
24 1356	37.698	36.633	31.716
25 1346	35.256	31.106	29.877
26 1345	47.016	42.269	40.199
34 1256	24.833	22.592	20.694
35 1246	28.794	25.927	23.390
36 1245	50.193	48.021	44.561
45 1236	30.629	28.950	27.092
46 1235	51.500	50.153	47.510
56 1234	56.080	55.390	51.521
123 456	56.661	55.474	54.345
124 356	54.402	52.666	52.044
125 346	50.653	48.953	47.993
126 345	28.957	26.470	22.905
134 256	47.675	46.956	46.154
135 246	47.279	43.109	40.016
136 245	34.237	29.400	26.527
145 236	47.331	42.957	41.199
146 235	35.340	34.072	30.970
156 234	39.487	38.426	36.880

TABLE III. The comparison of the violation of the separability condition for the six-partite state considered in Ref. [8]. Provided that the separability condition can be so strongly violated for all bipartitions it is absolutely unnecessary to test other kinds of separability, i.e. kinds with partitions of modes into three or more groups.

be violated with $s = 3.65$. The corresponding probability $\mathbf{P}(3.65) < 3 \cdot 10^{-4}$ is much smaller and gives a strong confidence that the state is 1,10|23456789-entangled. The vectors are

$$\begin{aligned}
\mathbf{h} = & (-0.65, -1.22, -0.21, 0.01, 0.365, \\
& -0.32, 0.25, 0.28, -0.88, -0.94), \\
\mathbf{g} = & (0.22, -1.04, -0.12, -0.04, 0.42, \\
& -0.34, 0.22, 0.30, -0.56, 0.73).
\end{aligned} \tag{B4}$$

The violations of other kinds of biseparability are all larger than 3, so the standard three-sigma test is passed for all bipartitions. The violations reported in Ref. [8]

show some strange behavior — the violation for full separability is smaller than violations of some more coarse kinds. But this may be an artifact of an implementation of the genetic optimization algorithm.

Up to now it has been shown that the four-partite state with the covariance matrix (B1) is not separable for any fixed kind of separability. We demonstrate that this state is genuine entangled. To do this we need to find a pair of matrices X and P that simultaneously violate the inequalities (31) for all bipartitions. First, we have tried to violate these inequalities with rank-one matrices. It took nearly one day, but we were able to find a pair of vectors

$$\begin{aligned} \mathbf{h} &= (0.31, -1.93, -0.17, -0.18) \\ \mathbf{g} &= (0.30, 1.48, -0.57, -0.53), \end{aligned} \quad (\text{B5})$$

that violate the conditions (31) for all seven bipartitions, and the minimal violation is 3.15 (for the bipartition 1|234). The corresponding probability $\mathbf{P}(3.15) = 1.6 \cdot 10^{-3}$ is relatively small to conclude that the state under study is genuine entangled.

The approach with a simpler conditions works but it takes a lot of time and it just marginally passes the three-sigma test. Using the general matrices we can do better. The sketch of our approach is as follows. We use a variant of the steepest gradient method. According to this method, to optimize a convex function one has to go in the direction opposite to the gradient of the function. Here we have several functions to be optimized at once, and each has its own gradient. We start by generating a pair of random positive definite matrices X and P and compute the gradients of all seven target functions. If the directions of these gradients are not strongly scattered then we can take the average of the gradients, go in the opposite direction and still improving all our functions simultaneously. If the gradients point into nearly opposite directions then we cannot proceed this way, so we stop and generate a new random pair of matrices. We do this until we find proper matrices X and P or give up after some prescribed number of attempts. Following this approach, in a few hours we found the following pair of matrices for the four-partite state with the covariance matrix (B1):

$$X = \begin{pmatrix} 0.39234 & -0.20267 & 0.24691 & 0.30527 \\ -0.20267 & 0.88526 & 0.09450 & 0.09080 \\ 0.24691 & 0.09450 & 0.58391 & 0.20795 \\ 0.30527 & 0.09080 & 0.20795 & 0.39504 \end{pmatrix}$$

$$P = \begin{pmatrix} 0.22992 & -0.13140 & -0.00477 & -0.11723 \\ -0.13140 & 0.52598 & -0.32316 & -0.16699 \\ -0.00477 & -0.32316 & 0.39949 & 0.06971 \\ -0.11723 & -0.16699 & 0.06971 & 0.31242 \end{pmatrix}$$

For these matrices we have $\mathcal{G} = 1.47484$ and $\sigma = 0.01947$. The bound $\mathcal{B}_{\mathcal{I}}(X, P)$ for different bipartitions is presented below. The elements of the matrices that were optimized over are highlighted. For the bipartition 1|234

the maximum is attained at

$$X' = \begin{pmatrix} 0.39234 & -0.10873 & 0.158136 & 0.116524 \\ -0.10873 & 0.88526 & 0.09450 & 0.09080 \\ 0.158136 & 0.09450 & 0.58391 & 0.20795 \\ 0.116524 & 0.09080 & 0.20795 & 0.39504 \end{pmatrix}$$

$$P' = \begin{pmatrix} 0.22992 & -0.11914 & 0.113758 & 0.083761 \\ -0.11914 & 0.52598 & -0.32316 & -0.16699 \\ 0.113758 & -0.32316 & 0.39949 & 0.06971 \\ 0.083761 & -0.16699 & 0.06971 & 0.31242 \end{pmatrix},$$

and is equal to $\mathcal{B}_{1|234}(X, P) = 1.65474$. For the bipartition 2|134 at

$$X' = \begin{pmatrix} 0.39234 & -0.07310 & 0.24691 & 0.30527 \\ -0.07310 & 0.88526 & -0.03586 & -0.01993 \\ 0.24691 & -0.03586 & 0.58391 & 0.20795 \\ 0.30527 & -0.01993 & 0.20795 & 0.39504 \end{pmatrix}$$

$$P' = \begin{pmatrix} 0.22992 & -0.05400 & -0.00477 & -0.11723 \\ -0.05400 & 0.52598 & -0.01432 & 0.02340 \\ -0.00477 & -0.01432 & 0.39949 & 0.06971 \\ -0.11723 & 0.02340 & 0.06971 & 0.31242 \end{pmatrix},$$

and is equal to $\mathcal{B}_{2|134}(X, P) = 1.66193$. For the bipartition 3|124 at

$$X' = \begin{pmatrix} 0.39234 & -0.20267 & 0.22149 & 0.30527 \\ -0.20267 & 0.88526 & -0.01671 & 0.09080 \\ 0.22149 & -0.01671 & 0.58391 & 0.24154 \\ 0.30527 & 0.09080 & 0.24154 & 0.39504 \end{pmatrix}$$

$$P' = \begin{pmatrix} 0.22992 & -0.13140 & 0.02836 & -0.11723 \\ -0.13140 & 0.52598 & -0.07629 & -0.16699 \\ 0.02836 & -0.07629 & 0.39949 & 0.12842 \\ -0.11723 & -0.16699 & 0.12842 & 0.31242 \end{pmatrix},$$

and is equal to $\mathcal{B}_{3|124}(X, P) = 1.56935$. For the bipartition 4|123 at

$$X' = \begin{pmatrix} 0.39234 & -0.20267 & 0.24691 & 0.15483 \\ -0.20267 & 0.88526 & 0.09450 & 0.04047 \\ 0.24691 & 0.09450 & 0.58391 & 0.23966 \\ 0.15483 & 0.04047 & 0.23966 & 0.39504 \end{pmatrix}$$

$$P' = \begin{pmatrix} 0.22992 & -0.13140 & -0.00477 & 0.05094 \\ -0.13140 & 0.52598 & -0.32316 & -0.05608 \\ -0.00477 & -0.32316 & 0.39949 & 0.12953 \\ 0.05094 & -0.05608 & 0.12953 & 0.31242 \end{pmatrix},$$

and is equal to $\mathcal{B}_{4|123}(X, P) = 1.63974$. For the bipartition

tion 12|34 at

$$X' = \begin{pmatrix} 0.39234 & -0.20267 & 0.19766 & 0.11649 \\ -0.20267 & 0.88526 & -0.02260 & 0.03156 \\ 0.19766 & -0.02260 & 0.58391 & 0.20795 \\ 0.11649 & 0.03156 & 0.20795 & 0.39504 \end{pmatrix}$$

$$P' = \begin{pmatrix} 0.22992 & -0.13140 & 0.11949 & 0.06522 \\ -0.13140 & 0.52598 & -0.02001 & 0.02362 \\ 0.11949 & -0.02001 & 0.39949 & 0.06971 \\ 0.06522 & 0.02362 & 0.06971 & 0.31242 \end{pmatrix},$$

and is equal to $\mathcal{B}_{12|34}(X, P) = 1.81056$. For the bipartition 13|24 at

$$X' = \begin{pmatrix} 0.39234 & -0.10013 & 0.24691 & 0.12997 \\ -0.10013 & 0.88526 & -0.03695 & 0.09080 \\ 0.24691 & -0.03695 & 0.58391 & 0.25436 \\ 0.12997 & 0.09080 & 0.25436 & 0.39504 \end{pmatrix}$$

$$P' = \begin{pmatrix} 0.22992 & -0.07273 & -0.00477 & 0.06225 \\ -0.07273 & 0.52598 & -0.05209 & -0.16699 \\ -0.00477 & -0.05209 & 0.39949 & 0.17734 \\ 0.06225 & -0.16699 & 0.17734 & 0.31242 \end{pmatrix},$$

and is equal to $\mathcal{B}_{13|24}(X, P) = 1.74993$. For the biparti-

tion 14|23 at

$$X' = \begin{pmatrix} 0.39234 & -0.11435 & 0.18571 & 0.30527 \\ -0.11435 & 0.88526 & 0.09450 & -0.02360 \\ 0.18571 & 0.09450 & 0.58391 & 0.25307 \\ 0.30527 & -0.02360 & 0.25307 & 0.39504 \end{pmatrix}$$

$$P' = \begin{pmatrix} 0.22992 & -0.09531 & 0.05497 & -0.11723 \\ -0.09531 & 0.52598 & -0.32316 & -0.02171 \\ 0.05497 & -0.32316 & 0.39949 & 0.12643 \\ -0.11723 & -0.02171 & 0.12643 & 0.31242 \end{pmatrix},$$

and is equal to $\mathcal{B}_{14|23}(X, P) = 1.56114$. The smallest number among these maximums is the last one, 1.56114, so we have

$$\frac{\mathcal{B}_{\mathcal{I}}(X, P) - \mathcal{G}(X, P)}{\sigma(X, P)} \geq s_0 = 4.43199$$

for all bipartitions \mathcal{I} simultaneously. The corresponding probability is $\mathbf{P}(s_0) < 10^{-5}$, which is almost two orders of magnitude smaller than for the vectors \mathbf{h} and \mathbf{g} we found before, so one can be pretty sure that the state under study is genuine entangled.

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